Hence $c_0 = d_0 = 0$, which results in specific conditions imposed on the functions $f_1(z)$ and $f_2(z)$.

As an illustration, let us consider the case when

$$f_1(z) = h_1 + h_2 z^2, \qquad f_2 = k_0 + k_1 z + k_2 z^2$$

(h_1, h_2, k_0, k_1, k_2 are constants). Then we obtain from (4.7)

$$c_n = (-1)^n \frac{2h_2}{n\omega}, \qquad d_n = \frac{2}{n\omega l}\{[(-1)^n - 1] k_1 + (-1)^n l k_2\} \tag{4.8}$$

$$h_1 = -\frac{l^3 h_2}{3}, \qquad k_0 = -l\left(\frac{k_1}{2} + \frac{l^2 k_2}{3}\right) \tag{4.9}$$

The relationships (4.9) impose constraints on the coefficients h and k.

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ON SOME PROPERTIES OF EQUATIONS OF A MODEL OF COUPLED TERMOPLASTICITY

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Within the scope of models of elastic-plastic media, without taking account of thermal effects, the rates of change in the stresses are determined uniquely by means of a given state of stress and strain rates [1]. The constraint which should be imposed on a coupled thermoplasticity model so that the mentioned property would also exist in this case is considered herein. It is shown for the simplest coupled thermoplasticity model, that when heat conduction is neglected, there exists a domain of states of stress for which the system of plastic flow equations is not evolutionary, and also a domain of states of stress for which shock formation occurs from smooth initial conditions (reversing of simple waves). These properties can also be interpreted as the properties of an uncoupled plasticity model with a nongradient plastic flow law. An exam-

ple of an unstable solution is presented for the complete system of plastic flow equations taking account of heat conduction.

1. The simplest model of coupled thermoplasticity is obtained under the following assumptions on the free energy F, uncompensated heat dq', the flow surface f and its associated law:

$$\rho_0 F = A^{ijkl}\varepsilon_{ij}{}^e\varepsilon_{kl}{}^e + \alpha^{ij}\varepsilon_{ij}{}^e (T - T_0) - \frac{m}{2}(T - T_0)^2 - \rho_0 s_0 (T - T_0)$$

$$dq' = \frac{1}{\rho_0}\sigma^{ij}\, d\varepsilon_{ij}{}^p, \quad f = f(\sigma_{ij}, T) = 0, \quad d\varepsilon_{ij}{}^p = d\lambda \frac{\partial f}{d\sigma^{ij}}, \quad d\lambda \geqslant 0 \qquad (1.1)$$

Here A^{ijkl}, α^{ij}, m, T_0, ρ_0, s_0 are constants, and all the notation are standard. The displacements and strains are considered small. For such a model [2]

$$\sigma^{ij} = A^{ijkl}\varepsilon_{kl}{}^e + \alpha^{ij}(T - T_0), \quad s = -\frac{1}{\rho_0}\alpha^{ij}\varepsilon_{ij}{}^e + \frac{m}{\rho_0}(T - T_0) + s_0 \qquad (1.2)$$

We obtain a necessary condition for the existence of the solution to the Cauchy problem for the system of equations of such a model. To do this, let us consider the problem of determining $\partial\sigma_{ij} / \partial t$, $\partial T / \partial t$ at the instant $t = t_0$ for given σ_{ij}, T and velocities v_i. It is assumed that the initial values of σ_{ij}, T satisfy the condition $f(\sigma_{ij}, T) = 0$, since otherwise $\partial\sigma_{ij} / \partial t$ and $\partial T / \partial t$ are determined uniquely from the equations of elasticity theory.

If plastic deformation should occur at the instant t_0 then we obtain the system to determine $(\partial\sigma_{ij} / \partial t)_p$, $(\partial T / \partial t)_p$, $(\partial\lambda / \partial t)_p$

$$\frac{\partial f}{\partial\sigma_{ij}}\left(\frac{\partial\sigma_{ij}}{\partial t}\right)_p + \frac{\partial f}{\partial T}\left(\frac{\partial T}{\partial t}\right)_p = 0$$

$$\left(\frac{\partial\sigma^{ij}}{\partial t}\right)_p = A^{ijkl}e_{kl}{}^\circ - \left(\frac{\partial\lambda}{\partial t}\right)_p A^{ijkl}\frac{\partial f}{\partial\sigma^{kl}} + \alpha^{ij}\left(\frac{\partial T}{\partial t}\right)_p \qquad (1.3)$$

$$-\alpha^{ij}\left[e_{ij}{}^\circ - \left(\frac{\partial\lambda}{\partial t}\right)_p \frac{\partial f}{\partial\sigma^{ij}}\right] + m\left(\frac{\partial T}{\partial t}\right)_p = \frac{1}{T_0}\frac{\partial q^e}{\partial t} + \frac{1}{T_0}\sigma_{ij}\frac{\partial f}{\partial\sigma_{ij}}\left(\frac{\partial\lambda}{\partial t}\right)_p$$

from (1.1), (1.2), the relationship

$$e_{ij}{}^\circ \equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \frac{\partial\varepsilon_{ij}{}^e}{\partial t} + \frac{\partial\varepsilon_{ij}{}^p}{\partial t}$$

and the equations of the second law of thermodynamics.

If elastic deformation occurs at the instant $t = t_0$, then we find analogously for $(\partial\sigma_{ij} / \partial t)_e$, $(\partial T / \partial t)_e$

$$\left(\frac{\partial\sigma^{ij}}{\partial t}\right)_e = A^{ijkl}e_{kl}{}^\circ + \alpha^{ij}\left(\frac{\partial T}{\partial t}\right)_e, \quad -\alpha^{ij}e_{ij}{}^\circ + m\left(\frac{\partial T}{\partial t}\right)_e = \frac{1}{T_0}\frac{\partial q^e}{\partial t} \qquad (1.4)$$

Discarding degeneration cases, it can be considered that each of the systems (1.3), (1.4) has exactly one solution. Moreover, in order for the solution (1.3) to be admissible in the elastic-plastic problem, it must satisfy the inequality $(\partial\lambda / \partial t)_p \geqslant 0$, and the solution (1.4) must satisfy

$$\left(\frac{\partial f}{\partial t}\right)_e \equiv \frac{\partial f}{\partial\sigma_{ij}}\left(\frac{\partial\sigma_{ij}}{\partial t}\right)_e + \frac{\partial f}{\partial T}\left(\frac{\partial T}{\partial t}\right)_e \leqslant 0$$

It is assumed here that the interior domain relative to the flow surface in σ_{ij}, T space corresponds to $f < 0$ and the exterior domain to $f > 0$. From (1.3), (1.4) we find

$$\left(\frac{\partial f}{\partial t}\right)_e = \left(\frac{\partial \lambda}{\partial t}\right)_p M,\ M \equiv A^{ijkl}\frac{\partial f}{\partial \sigma^{ij}}\frac{\partial f}{\partial \sigma^{kl}} + \frac{1}{mT_0}\left(\alpha^{ij}\frac{\partial f}{\partial \sigma^{ij}} + \frac{\partial f}{\partial T}\right) \times \left(T_0\alpha^{ij}\frac{\partial f}{\partial \sigma^{ij}} - \sigma^{ij}\frac{\partial f}{\partial \sigma^{ij}}\right)$$

In order for $\partial\sigma_{ij} / \partial t,\ \partial T / \partial t$ to be determined uniquely for any initial data $\sigma_{ij},\ T,\ v_i,\ f(\sigma_{ij},\ T) = 0$, it is necessary that $M \geqslant 0$. Otherwise, for the state in which $M(\sigma_{ij},\ T) < 0$, initial values of v_i can be indicated such that

$$\left(\frac{\partial f}{\partial t}\right)_e = A^{ijkl}\frac{\partial f}{\partial \sigma^{ij}} e_{kl}^\circ + \frac{1}{mT_0}\left(\alpha^{ij}\frac{\partial f}{\partial \sigma^{ij}} + \frac{\partial f}{\partial T}\right)\left(\frac{\partial q^e}{\partial t} + T_0\alpha^{ij}e_{ij}^\circ\right) > 0$$

$$\left(\frac{\partial \lambda}{\partial t}\right)_p < 0$$

and therefore, it is impossible to construct a plastic or elastic solution for such an initial state. In this case initial data can also be given so that $(\partial f / \partial t)_e < 0,\ (\partial\lambda / \partial t)_p > 0$, and therefore, $\partial\sigma_{ij} / \partial t,\ \partial T / \partial t$ are not determined uniquely at the instant t_0. If $M > 0$, then $(\partial f / \partial t)_e$ and $(\partial\lambda / \partial t)_p$ have the same signs and vanish only simultaneously (in this latter case the solutions (1.3) and (1.4) agree). In this case $\partial\sigma_{ij} / \partial t$ and $\partial T/\partial t$ are determined uniquely. Therefore, when constructing models of elastic-plastic media, compliance with the inequality $M \geqslant 0$ should be assured.

Let us note some particular cases. Let the form $A^{ijkl}\varepsilon_{ij}\varepsilon_{kl}$ be positive definite, $m > 0,\ \alpha^{ij} = -\alpha g^{ij}$. If the equation of the flow surface is taken in the Mises form ${}^1/_2\sigma_{ij}'\sigma_{ij}' = k^2(T),\ \sigma_{ij}' \equiv \sigma_{ij} - {}^1/_3\sigma_{kk}\delta_{ij}$, then

$$M = A^{ijkl}\frac{\partial f}{\partial \sigma^{ij}}\frac{\partial f}{\partial \sigma^{kl}} + \frac{k^2}{mT_0}\frac{dk^2}{dT}$$

The condition $M > 0$ imposes a constraint on the assumption of the rate of diminution of the yield point as the temperature rises. If the flow surface doesn't depend on temperature, then

$$M = A^{ijkl}\frac{\partial f}{\partial \sigma^{ij}}\frac{\partial f}{\partial \sigma^{kl}} + \frac{\alpha^2}{m}\left(\frac{\partial f}{\partial \sigma_{kk}}\right)^2 + \frac{\alpha}{mT_0}\frac{\partial f}{\partial \sigma_{kk}}\frac{\partial f}{\partial \sigma_{ij}}\sigma_{ij}$$

Since $\sigma_{ij}\partial f/\partial\sigma_{ij} \geqslant 0$ for media with a convex flow surface, containing the point $\sigma_{ij} = 0$ internally, in order to satisfy the condition $M \geqslant 0$ it is sufficient that $\alpha\ \partial f / \partial p \leqslant 0,\ p \equiv -{}^1/_3\ \sigma_{kk}$. In particular, this is satisfied for the propagated condition ${}^1/_2\sigma_{ij}\sigma_{ij} = F(p),\ dF / dp \geqslant 0$ for $\alpha > 0$.

In the case of uncoupled plasticity theory $(\alpha_{ij} = 0,\ \partial f / \partial T = 0)$ $M = A^{ijkl}(\partial f / \partial\sigma_{ij})(\partial f / \partial\sigma_{kl})$, we arrive at the known result obtained in [1] for flow surfaces with singularities and for hardening media also.

2. If the medium is isotropic, the expression for the free energy becomes

$$\rho_0 F = {}^1/_2\lambda(\varepsilon_{kk}^e)^2 + \mu\varepsilon_{ij}^e\varepsilon_{ij}^e - \alpha(3\lambda + 2\mu)(T - T_0)\varepsilon_{kk}^e - {}^1/_2 m(T - T_0)^2 - \rho_0 s_0(T - T_0)$$

Let us examine the motion of such a medium by plane waves for $v_3 = 0,\ \sigma_{13} = \sigma_{23} = \sigma_{22} - \sigma_{33} = 0$ (compliance with these equalities at the initial instant is sufficient). The Mises condition ${}^1/_2\ \sigma_{ij}'\sigma_{ij}' = k^2$ then reduces to ${}^3/_4\ (\sigma_{11} + p)^2 + \sigma_{12}^2 = k^2$. Let us introduce the new variable θ

$$\sigma_{12} = k\sin\theta,\ {}^1/_2\sqrt{3}(\sigma_{11} + p) = k\cos\theta,\quad 0 \leqslant \theta < 2\pi$$

In the case under consideration, the system (1.1), (1.2) and the equations of motion describing the plastic flow of the medium reduce to

$$\rho_0 \frac{\partial v_1}{\partial t} = -\frac{\partial p}{\partial x} - \frac{2k \sin\theta}{\sqrt{3}} \frac{\partial \theta}{\partial x}, \qquad \rho_0 \frac{\partial v_2}{\partial t} = k \cos\theta \frac{\partial \theta}{\partial x} \tag{2.1}$$

$$\frac{\partial v_1}{\partial x} = -\frac{1}{K} \frac{\partial p}{\partial t} + 3\alpha \frac{\partial T}{\partial t} \qquad \left(K \equiv \lambda + \frac{2}{3}\mu\right) \tag{2.2}$$

$$\frac{\sin\theta}{\sqrt{3}} \frac{\partial v_1}{\partial x} - \frac{\cos\theta}{2} \frac{\partial v_2}{\partial x} + \frac{k}{2\mu} \frac{\partial \theta}{\partial t} = 0 \tag{2.3}$$

$$k \frac{\partial \lambda}{\partial t} = \frac{\cos\theta}{\sqrt{3}} \frac{\partial v_1}{\partial x} + \frac{\sin\theta}{2} \frac{\partial v_2}{\partial x} \geqslant 0 \tag{2.4}$$

We write down the equation of the second law of thermodynamics governing the heat influx by the Fourier law

$$c \frac{\partial T}{\partial t} = 3\alpha \frac{\partial p}{\partial t} + \frac{2k^2}{T_0} \frac{\partial \lambda}{\partial t} + \frac{\varkappa}{T_0} \frac{\partial^2 T}{\partial x^2} \qquad (c \equiv m + 9\alpha^2 K) \tag{2.5}$$

Let us first consider a corresponding ideal system describing the large-scale phenomena (2.1)–(2.5). The passage to the ideal system is accomplished by neglecting terms with higher derivatives, and in this case is equivalent to considering adiabatic processes or to the assumption $\varkappa = 0$. In this case, by using (2.4), (2.5), Eq. (2.2) can be given the form

$$\frac{\partial v_1}{\partial x} = -\frac{1}{K_1} \frac{\partial p}{\partial t} + \frac{k_1}{k} \left(\frac{\cos\theta}{\sqrt{3}} \frac{\partial v_1}{\partial x} + \frac{\sin\theta}{2} \frac{\partial v_2}{\partial x} \right), \quad \frac{1}{K_1} \equiv \frac{1}{K} - \frac{9\alpha^2}{c},$$
$$k_1 \equiv \frac{6\alpha k^2}{c T_0} \tag{2.6}$$

Equations (2.1) – (2.3), (2.6) form a closed ideal system of plastic flow equations. Let us note that this system can be considered as the equations of uncoupled plasticity theory for a medium with the same flow surface but with a nongradient plastic flow law. The system under consideration differs from the equations of uncoupled plasticity theory with a gradient law only by the insertion of K_1 instead of K and by the presence of the last member in (2.6). The ratios k_1 / k and $(K_1 - K) / K$ are on the order of 10^{-3} for steel, say, hence, neglecting thermal effects is justified in many cases. But as is shown below, for plane waves in which the tangential stresses on areas parallel to the front are close to k, taking them into account is of value, in principle.

Indeed, let us consider the characteristic equation of the system (2.1) – (2.3), (2.6)

$$D(C) \equiv (\rho_0 C^2)^2 - \beta\rho_0 C^2 + K_1\mu \left(\cos\theta - \frac{k_1}{k\sqrt{3}}\right) \cos\theta = 0 \tag{2.7}$$

$$\beta \equiv K_1 + \frac{4\mu}{3} \sin^2\theta + u \cos^2\theta - \frac{k_1 K_1}{k\sqrt{3}} \cos\theta$$

Here C is the characteristic velocity. Equation (2.7) has two pairs of roots $\pm C_+$, $\pm C_-$ $(C_+^2 > C_-^2)$. If $\alpha = 0$, then the roots C_+°, C_-° of (2.7) are real, where $C_+^\circ \geqslant \sqrt{\mu / \rho_0}$, and C_-° vanishes for $\theta = \pi/2$ $(3\pi / 2)$ [3]. In case $\alpha \neq 0$, the changes in the coefficients of (2.7) which are small compared to K cause small deviations in the quantities C, $dC / d\theta$ from C°, $dC^\circ / d\theta$. However, these deviations can turn out to be substantial near points in which $C^\circ = 0$ or $dC^\circ / d\theta = 0$. Thus, from (2.7) we obtain that $\rho_0 C_-^2 < 0$ for

$$\theta_* < \theta < \pi / 2 \qquad (3\pi / 2 < \theta < 2\pi - \theta_*), \quad \theta_* = \arccos\, k_1 / k\sqrt{3}$$

The characteristic velocities become imaginary. The sign of $dC_{\pm}^{\circ} / d\theta$ agrees with the sign of $\pm \sin\theta\cos\theta$. Differentiating (2.7) with respect to θ we find

$$\operatorname{sgn}\frac{d(\rho_0 C_+{}^2)}{d\theta} =$$

$$\pm\operatorname{sgn}\left[\rho_0 C_{\pm}{}^2\left(\frac{2\mu\cos\theta}{3} + K_1\cos\theta_*\right)\sin\theta + K_1\mu\left(2\cos\theta - \cos\theta_*\right)\sin\theta\right]$$

Hence, it is seen that $d(\rho_0 C_-{}^2) / d\theta$ changes sign for $\theta = \theta_1 \in (\theta_*, \pi/2)$, $\theta = 0$, $\theta = \pi$ and at their symmetric points relative to $\theta = \pi$; $d(\rho_0 C_+{}^2)/d\theta$ changes sign for $\theta = \theta_2 \in (\pi/2, \ \pi - \arccos\ [3K_1 k_1 / 2\sqrt{3}k\mu])$, $\theta = 0$, $\theta = \pi$ and at their symmetric points.

The dependence of the characteristic velocities on θ is illustrated in Fig. 1 (also presented for comparison are corresponding dependencies for $\alpha = 0$). The quantity $\rho_0 C_+{}^2$ reaches the maximum $K_1 + 4\mu/3$ for $\theta = \theta_2$, the minimum $-K_1(1 - k_1/k\sqrt{3})$ for $\theta = 0$ and has a local minimum $K_1(1 + k_1/k\sqrt{3})$ for $\theta = \pi$. For $\pi < \theta < 2\pi$ the graphs are symmetric to those presented relative to the line $\theta = \pi$.

An essential singularity of the ideal system under consideration is the presence of the imaginary characteristic velocity. The Cauchy problem for such a system is formulated incorresctly [4]. This fact can be interpreted as either the result of selecting a nongradient flow law in the uncoupled model, or the result of neglecting the heat conductivity in the coupled model. If the heat conductivity is taken into account, then the system, as compared with an ideal system, loses a pair of imaginary characteristic velocities, and the remaining characteristic velocities are real.

Let us show that the system (2.1) – (2.5) is evolutionary, i.e. that $\operatorname{Im}\omega(l)$ has an upper bound, where $\omega(l)$ is the root of the dispersion equation of the system corresponding to real l. The dispersion equation is

$$-i\omega D\left(\frac{\omega}{l}\right) + \frac{\varkappa K_1}{cT_0K} l^2 D^\circ\left(\frac{\omega}{l}\right) = 0 \qquad (2.8)$$

Here D is defined by (2.7) and D° denotes D for $\alpha = 0$, $D^\circ(C) = 0$ is the characteristic equation of the uncoupled model. Equation (2.8) is algebraic, hence, unboundedness of $\operatorname{Im}\omega$ is possible only for $\omega \to \infty$. Simultaneously there should be $l \to \infty$. If ω / l is hence bounded, then

$$D^\circ\left(\frac{\omega}{l}\right) = -\frac{1}{l}\left[\frac{icKT_0}{\varkappa K_1}\frac{\omega}{l} D\left(\frac{\omega}{l}\right)\right]$$

where the expression in the bracket is bounded, and therefore $\omega / l = C_{\pm}^{\circ} + O(l^{-1})$, $\operatorname{Im}\omega = O(1)$. If $\omega / l \to \infty$, then $D(\omega/l) / D^\circ(\omega/l) \to 1$ and it follows from (2.8) that

$$-\frac{i\omega}{l^2} = \frac{\varkappa K_1}{cT_0K} + o(1), \qquad \operatorname{Im}\omega \to -\infty$$

Therefore, the system (2.1) – (2.5) is evolutionary.

It is impossible to seek a perturbation of the form $Ae^{i(lx-\omega t)}$ in investigating the stability of constant solutions of the system (2.1) – (2.5) (for them, in particular, $\partial\lambda / \partial t = 0$) since for such perturbations there are segments on which $(\partial\lambda / \partial t)_p > 0$ and segments with $(\partial\lambda / \partial t)_p < 0$ at the initial instant. The flow picture will have a complex structure of alternating plastic and elastic domains at subsequent instants, which abut on the appropriate segments in conformity with Sect. 1. However, such a method is applicable to an investigation of the stability of the plastic solutions in which $\partial\lambda / \partial t \geqslant \psi > 0$ relative to the small perturbations not reducing σ_{ij}, T from the flow surface.

For example, let us consider the stability of the simplest non-constant solution of the system (2.1) – (2.5)

$$\theta = \theta_0 = \text{const}, \quad p = p_0 + p_1 t, \quad v_1 = {}^1/_2 A \sqrt{3} \operatorname{ctg} \theta_0 x$$
$$v_2 = Ax, \quad T = \frac{1}{6\alpha}\left[A\sqrt{3}\operatorname{ctg}\theta_0 t 2 p_1 t / K + \left(bA + \frac{cT_0}{\varkappa K_1} p_1\right)x^2 + H\right]$$
$$\frac{\partial\lambda}{\partial t} = \frac{A}{2k\sin\theta_0} > 0, \qquad b \equiv \frac{\sqrt{3}\,(cT_0\cos\theta_0 - 2\sqrt{3}\alpha k)}{2\sin\theta_0\,\varkappa} \tag{2.9}$$

Here $A > 0$, H, p_1 are arbitrary constants. It is possible to consider (2.9) for $-\infty < x < \infty$ as a solution of the Cauchy problem with appropriate initial data for v_1, v_2, p, θ or for $-L \leqslant x \leqslant L$ as the solution of a boundary value problem with the boundary conditions

$$v_1 = \pm\frac{1}{2}AL\sqrt{3}\operatorname{ctg}\theta_0, \quad v_2 = \pm AL, \quad \frac{\partial T}{\partial x} = \pm(3\alpha)^{-1}\left(bA + \frac{c}{\varkappa K_1}T_0 p_1\right)L$$
$$\text{for} \quad x = \pm L$$

To investigate the stability of this solution, let us linearize the system (2.1) – (2.5) around it. The system coefficients depend only on θ hence, the linearized system has the same form as the system (2.1) – (2.5), but θ_0 must be substituted into the coefficients in place of θ. Moreover, a new term originates during linearization, namely: $A\,(2\sin\theta_0)^{-1}\,\theta$ is added in the left side of (2.3). Small additions to the solution (2.9) are understood to be θ, v_1, v_2, p, T.

An analogous change occurs upon composing the determinant $D^*(\omega, l)$ to compute the dispersion equation of the linearized system: a term $ik\omega / 2\mu + A\,(2\sin\theta_0)^{-1}$ appears in place of $ik\omega / 2\mu$ in the row corresponding to (2.3) and in the column corresponding to θ. Therefore, the equation $D^*(\omega, l) = 0$ contains, besides all the terms of the dispersion equation of the system (2.1) – (2.5), additional terms generated by the addition of $A\,(2\sin\theta_0)^{-1}$. Hence, in addition to all the additional terms of the form $aA\,(2\sin\theta_0)^{-1}\,\omega^p l^q$ there is the term $aik\,(2\mu)^{-1}\omega^{p+1}l^q$ which is also present in the dispersion equation of the system (2.1) – (2.5). For $\omega \to \infty$ the additional term can be neglected as compared with $aik\,(2\mu)^{-1}\,\omega^{p+1}l^q$, hence the equation $D^*(\omega, l) = 0$ cannot, exactly as (2.8), have the roots $\omega(l)$ with $\operatorname{Im}\omega$ without an upper bound. Therefore, the growth rate of the small perturbations of the solution (2.9) is bounded.

The dispersion equation of the linearized system is

$$c\rho_0^2 K_1^{-1}\Omega^5 + \rho_0^2\left(\frac{\varkappa l^2}{KT_0} + \frac{cA_1}{K_1\sin\theta_0}\right)\Omega^4 + \rho_0 l^2\left(\frac{c\beta}{K_1} + \frac{\varkappa\rho_0 A_1}{kT_0\sin\theta_0}\right)\Omega^3 +$$
$$\rho_0 l^2\left[\frac{\varkappa\beta^\circ}{KT_0}l^2 + \frac{A_1}{\sin\theta_0}\left(\frac{6k\alpha\cos\theta_0}{\sqrt{3}\,T_0} - c\right)\right]\Omega^2 +$$
$$l^4\left[\frac{c\cos\theta_0}{\mu}\left(\cos\theta_0 - \frac{k_1}{k\sqrt{3}}\right) + \frac{\rho_0\varkappa A_1}{T_0\sin\theta_0}\right]\Omega + \tag{2.10}$$
$$\frac{\mu\varkappa\cos^2\theta_0}{T_0}l^6 = 0, \qquad \Omega \equiv -i\omega, \qquad A_1 = \frac{\mu A}{k}$$

where β is defined by (2.7), and β° denotes β for $\alpha = 0$. There are roots in the right half-plane for Eq. (2.10). Otherwise, all its coefficients would have the same sign, but the coefficient of Ω^5 is positive, and of Ω is negative if $\theta_* < \theta < \pi/2$ and A is sufficiently small.

Therefore, the equation $D^*(\omega, l) = 0$ has the roots $\omega = i\Omega$, $\operatorname{Im}\omega > 0$. Consequently, the solution (2.9) of the Cauchy problem for $-\infty < x < \infty$ is unstable. Equation (2.10) is invariant relative to replacement of l by $-l$, hence, the global instability of (2.9) as a solution of the boundary value problem for sufficiently large L [5] also follows from the existence of the roots ω, $\operatorname{Im}\omega > 0$.

Finally, let us consider the question of reversal of the simple waves of the system (2.1) – (2.3), (2.6). The equations describing simple waves, for which θ can be taken as the parameter (*), are

$$\rho_0 C \frac{dv_1}{d\theta} = \frac{dp}{d\theta} + \frac{2k}{\sqrt{3}} \sin\theta, \qquad -\rho_0 C \frac{dv_2}{d\theta} = k\cos\theta$$

$$\frac{dv_1}{d\theta} = \frac{k\sqrt{3}}{2\mu} \frac{\rho_0 C^2 - \mu\cos^2\theta}{\rho_0 C \sin\theta}$$

$$\frac{dv_1}{d\theta} = \frac{C}{K_1}\frac{dp}{d\theta} + \frac{k_1}{2k}\sin\theta \frac{dv_2}{d\theta} + \frac{k_1\cos\theta}{k\sqrt{3}} \frac{dv_1}{d\theta}$$

$$\frac{\partial\lambda}{\partial t} = \frac{\mu - \rho_0 C^2}{2\mu\rho_0 C^2} \operatorname{ctg}\theta \frac{\partial\theta}{\partial t} \tag{2.11}$$

where C is determined from the characteristic equation (2.7).

From (2.7) we find

$$\rho_0 C_-^2 \leqslant \mu, \qquad \rho_0 C_+^2 \geqslant K_1 \left(1 - \frac{k_1}{\sqrt{3}k}\right)$$

It is here assumed that $K/\mu > 4/3$, and therefore, $\rho_0 C_+^2 > \mu$ for sufficiently small k_1/k. Because of these inequalities it follows from (2.11) that the sign of $\partial\theta/\partial t$ agrees with the sign of $\operatorname{ctg}\theta$ in slow simple waves, and is opposite in fast waves. Taking this relationship and the sign of the derivative $d(\rho_0 C_\pm^2)/d\theta$ into account (see Fig. 1),

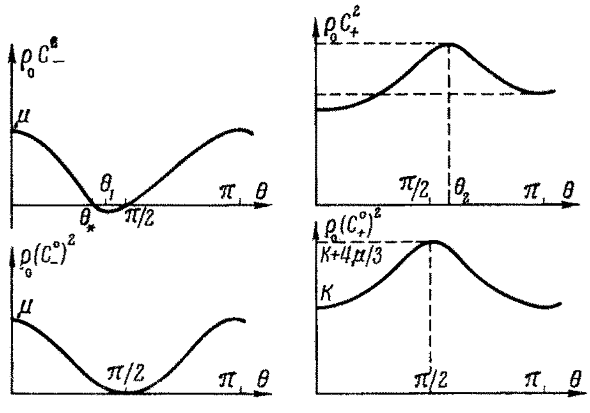

Fig. 1

*) Simple waves in which it is impossible to take θ as the parameter ($\theta = \text{const}$) are propagated without a change in shape.

we obtain that the condition for reversal of waves being propagated to the right $\partial C / \partial t > 0$ is satisfied for $\pi / 2 < \theta < \theta^2$ $(2\pi - \theta_2 < \theta < 3\pi / 2)$ for fast plastic waves. Therefore, taking account of thermal effects results in the need to consider jumps in the plastic domain in contrast to the uncoupled model [6]. In those cases when the jump is of sufficiently small intensity, for example, if it originates because of reversal of the simple wave and $\theta_2 - \pi / 2$ is small (this quantity is on the order of 10^{-3} for steel), the relationships between the quantities in the appropriate simple wave can be used as approximate conditions on the jump, and the rate of propagation of the discontinuity can approximately be considered equal to the average of the values of the appropriate characteristic velocities ahead of and behind the jump.

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ON THE HAMILTON-JACOBI METHOD FOR NONHOLONOMIC SYSTEMS

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The progress made in the theory of integration of equations of motion of holonomic systems naturally leads to attempts to extend the basic assumptions of this theory to nonholonomic systems, or at least to establish the conditions for their applicability to nonholonomic systems. Problems of this type were the subject of many papers by various authors. In particular, numerous attempts were made to extend the Hamilton-Jacobi method of integration to the nonholonomic systems (see [1]). Below we discuss the problems relevant to the latter problem.